Review of spanning sets and linear independence in $\mathbb{R}^n$

In these notes we review properties of spanning sets and linear independence in $\mathbb{R}^n$ from four different perspectives: linear systems of equations, ordered lists of vectors, matrices, and linear maps.

We use the following setup. Fix positive integers m and n. Let $a_{11}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn}$, and $b_1, \ldots, b_m$ be real numbers. We fix the a_{ij} numbers, but we sometimes vary the b_i numbers below. Let E denote the following linear system of equations in $x_1, \ldots, x_n$:

$$E : \begin{cases} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ & & \vdots & & & & \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

We also put the coefficients in a list of vectors in $\mathbb{R}^n$. Let

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, \vec{a}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}, \text{ and } \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Let $S = \{\vec{a}_1, \ldots, \vec{a}_n\}$. We also put the a_{ij} numbers in an $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Finally, we use these numbers to define a linear function. Let $L \colon \mathbb{R}^n \to \mathbb{R}^m$ be defined by $L(\vec{x}) = A\vec{x}$ for every $x \in \mathbb{R}^n$. These four objects (E, S, A, and L) are different perspectives on the same thing. In particular, we can characterize properties of S in terms of properties of the other objects.

First we characterize linear independence. Recall that S is linearly independent (by definition) if the only way to write

$$\vec{0} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

is if $x_1 = \cdots = x_n = 0$.

Fact. *The following are equivalent (any one implies any other):*

- *S is linearly independent.*
- *For every $\vec{b} \in \mathbb{R}^m$, $\vec{b}$ can be written as*

  $$\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

  for at most one choice of $x_1, \ldots, x_n$.
- *For every choice of $b_1, \ldots, b_m$, E has at most one solution.*
- *If $b_1 = \cdots = b_m = 0$, then E has exactly one solution.*
- *For every $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has at most one solution.*

- *The equation* $A\vec{x} = \vec{0}$ *has exactly one solution.*
- *The RREF of* A *has a pivot entry in every column.*
- *The rank of* A *is* n*.*
- *The nullspace of* A *is* $\{\vec{0}\}$*.*
- L *is an injective function (in other words* L *is a one-to-one function, or* L *does not smash different things together).*
- *The equation* $L(\vec{x}) = \vec{0}$ *has exactly one solution.*

The next fact characterizes being a spanning set. Recall that $\text{span}(S)$ is the set of all possible linear combinations of vectors from S, and we say that S spans $\mathbb{R}^m$ if $\text{span}(S) = \mathbb{R}^m$.

Fact. *The following are equivalent:*

- S *spans* $\mathbb{R}^m$*.*
- *For every* $\vec{b} \in \mathbb{R}^m$*,* $\vec{b}$ *can be written as*

$$\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

  for at least one choice of $x_1, \ldots, x_n$*.*
- *For all choices of* $b_1, \ldots, b_m$*, the system* E *is consistent.*
- *For every* $\vec{b} \in \mathbb{R}^m$*, the equation* $A\vec{x} = \vec{b}$ *is consistent.*
- *The RREF of* A *has no zero rows.*
- *The rank of* A *is* m*.*
- *The column space of* A *is* $\mathbb{R}^m$*.*
- L *is a surjective function (in other words* L *is an onto function, or* L *hits everything in* $\mathbb{R}^m$*).*

We put these facts together to characterize being a basis. Recall that S is a basis for $\mathbb{R}^m$ if and only if S is linearly independent and S spans $\mathbb{R}^m$.

Fact. *The following are equivalent:*

- S *is a basis for* $\mathbb{R}^m$*.*
- *For every* $\vec{b} \in \mathbb{R}^m$*,* $\vec{b}$ *can be written as*

$$\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

  for exactly one choice of $x_1, \ldots, x_n$*.*
- *For all choices of* $b_1, \ldots, b_m$*, the system* E *has exactly one solution.*
- $m = n$ *and the RREF of* A *is the* $m \times m$ *identity matrix.*

- *$m = n$ and the rank of A is m.*
- *$m = n$ and A is invertible.*
- *$m = n$ and A has nonzero determinant.*
- *L is an invertible function.*

It turns out it matters whether n is bigger than m, as we explain below.

Fact. *Suppose $n > m$ (this means there are more vectors in the list than entries in any one vector).*
Then S is linearly dependent.
However, S may or may not span $\mathbb{R}^m$.

To check whether S spans $\mathbb{R}^m$, put A in RREF. If there are no zero rows, then S spans $\mathbb{R}^m$.

Fact. *Suppose $m > n$ (this means there are more entries in one of the vectors than there are vectors in the list).*
Then S does not span $\mathbb{R}^m$.
However, S may or may not be linearly dependent.

To check whether S is linearly independent, put A in RREF. If every column has a pivot entry (in other words, the system has no free variables), then S is linearly independent.

Fact. *Suppose $m = n$.*
Then S is a basis if and only if it is linearly independent, if and only if it spans $\mathbb{R}^m$.
However, it is possible that S is not a basis.

To check whether S is a basis, take the determinant of A or row reduce A. If $\det(A) \neq 0$, S is a basis. If the RREF of A is the identity matrix, then S is a basis.